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ON THE CONTINUOUS DEPENDENCE OF A LINEAR ENCOUNTER GAME ON A PARAMETER

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E. G. AL'BREKHT and M. I. LOGINOV
(Sverdlovsk)

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We examine an encounter game problem for linear objects whose equations of motion are integrally continuous relative to a certain parameter. We establish sufficient conditions ensuring, in the regular case, the continuity of the game's value and the semicontinuity of the set of the players' optimal motions with respect to variations of this parameter.

1. Let the motions $y(t)$ of a pursuing object and $z(t)$ of a pursued object be described by the linear differential equations

$$y^{\cdot} = A^{(1)}(t, \lambda)\, y + B^{(1)}(t, \lambda)\, u, \quad u \in P \tag{1.1}$$

$$z^{\cdot} = A^{(2)}(t, \lambda)\, z + B^{(2)}(t, \lambda)\, v, \quad v \in Q \tag{1.2}$$

Here $y = \{y_1, \ldots, y_n\}$ and $z = \{z_1, \ldots, z_n\}$ are the phase coordinate vectors of the controlled objects; u and v are r-dimensional control vectors; P and Q are bounded, convex and closed sets describing the constraints on the players' controls; $A^{(j)}(t, \lambda)$ and $B^{(j)}(t, \lambda)$, $(j = 1,2)$ are matrices of appropriate dimensions, continuous in $t \in [t_0, \vartheta]$ and bounded in λ; $\lambda \in \Lambda$, Λ is some set of values of parameter λ, containing the limit point λ_0.

This paper's purpose is to study the encounter game problem (see [1], Sect. 7) for the objects (1.1) and (1.2) when the game's payoff is determined by the quantity $\gamma[\vartheta] = \|\{y[\vartheta]\}_m - \{z[\vartheta]\}_m\|$, where $\|x\|$ is the Euclidean norm of vector x and $\{x\}_m$ is the vector composed of the first m components of vector x. In the regular case we investigate the dependence on λ of the game's value and of the players' optimal motions, under the condition that the right-hand sides of Eqs. (1.1) and (1.2) are integrally continuous [2, 3] functions of parameter λ at the point λ_0. The presentation of the material relies on the concept of extremal construction and on the extremal aiming rule, justified in [1].

Definition 1.1. A strategy $U_\lambda \div U(t, y, z, \lambda)$ is said to be integrally semicontinuous in the parameter λ at the point λ_0 with respect to the matrix $B^{(1)}(t, \lambda)$ if: (1) for each $\lambda \in \Lambda$ the sets $U(t, y, z, \lambda)$ are convex, closed and upper-semicontinuous with respect to inclusion as t, y and z vary in a neighborhood of each possible position; (2) the matrix $B^{(1)}(t, \lambda)$, $t \in [t_0, \vartheta]$ is bounded and integrally continuous in parameter λ at the point λ_0; (3) for any family of vector functions $x(t, \lambda) = \{y(t, \lambda), z(t, \lambda)\}$, continuous in λ and such that the limit relation $\lim x(t, \lambda) = x(t, \lambda_0)$ is satisfied uniformly relative to $t \in [t_0, \vartheta]$ as $\lambda \to \lambda_0$, the set

$$\int_{t_0}^{t} B^{(1)}(\tau, \lambda)\, U(\tau, y(\tau, \lambda), z(\tau, \lambda), \lambda)\, d\tau$$

is upper-semicontinuous with respect to inclusion as the parameter λ varies in a neighborhood of point λ_0 for all $t \in [t_0, \vartheta]$.

The strategy $V_\lambda \div V(t, y, z, \lambda)$, integrally semicontinuous in parameter λ at point λ_0 with respect to matrix $B^{(2)}(t, \lambda)$, is defined similarly. The integrally semicontinuous strategies U_λ and V_λ are admissible for each $\lambda \in \Lambda$; therefore, for each $\lambda \in \Lambda$ Eqs. (1.1) and (1.2) possess [1, 4, 5] a family of motions $X(U_\lambda, V_\lambda; t_0, y^\circ, z^\circ, \lambda)$ consisting of the absolutely continuous solutions $x[t, \lambda] = \{y[t, \lambda], z[t, \lambda]\}$ of these equations, generated by the strategies U_λ and V_λ for an arbitrary initial position $\{t_0, y^\circ, z^\circ\}$. On the basis of the reasonings and results of [1—5] the validity of the following statement can be confirmed.

Theorem 1.1. Let the admissible strategies U_λ and V_λ be integrally semicontinuous in parameter λ at point λ_0 with respect to the matrices $B^{(1)}(t, \lambda)$ and $B^{(2)}(t, \lambda)$ respectively, and let the matrices $A^{(1)}(t, \lambda)$ and $A^{(2)}(t, \lambda)$ be bounded and integrally continuous in λ at point λ_0; then for any preselected number $\alpha > 0$ we can find a neighborhood $\Omega(\lambda_0)$ of point λ_0 such that:

1) For all $\lambda \in \Omega(\lambda_0)$ the families of Euler polygonal lines $X^{(\Delta)}(U_\lambda, V_\lambda; t_0, y^\circ, z^\circ, \lambda)$ lie in the α-neighborhood of the family $X^{(\Delta)}(U_{\lambda_0}, V_{\lambda_0}; t_0, y^\circ, z^\circ, \lambda_0)$. The neighborhood $\Omega(\lambda_0)$ can be chosen independent of the partitioning Δ and of the initial position $\{t_0, y^\circ, z^\circ\}$, from an arbitrary bounded region Γ in space $\{t, y, z\}$.

2) For all $\lambda \in \Omega(\lambda_0)$ the families of motions $X(U_\lambda, V_\lambda; t_0, y^\circ, z^\circ, \lambda)$ lie in the α-neighborhood of the family $X(U_{\lambda_0}, V_{\lambda_0}; t_0, y^\circ, z^\circ, \lambda_0)$. The neighborhood $\Omega(\lambda_0)$ can be chosen independent of the initial position $\{t_0, y^\circ, z^\circ\} \in \Gamma$.

2. We consider the controlled system

$$dx / d\tau = A(\tau, \lambda)x + B(\tau, \lambda)w, \quad w \in R \tag{2.1}$$

where $A(\tau, \lambda)$ and $B(\tau, \lambda)$ are matrices continuous in τ, bounded in $\lambda \in \Lambda$ and integrally continuous in parameter λ at point λ_0; R is a convex, closed and bounded set. Let $G(\vartheta, t, x, \lambda)$ be the attainability region of system (2.1) in the m-dimensional space $\{q\}$ of points $q = \{x\}_m$ from the state $\tau = t \geqslant t_0$ and $x(t) = x$ by the instant $\tau = \vartheta$. For each $\lambda \in \Lambda$ the attainability region $G(\vartheta, t, x, \lambda)$ is a convex, closed and bounded set whose support function $\rho[l, \vartheta, t, x, \lambda]$ is described by the equality

$$\rho[l, \vartheta, t, x, \lambda] = l'\{X[\vartheta, t, \lambda]x\}_m + \max_{w \in R} \int_t^\vartheta l'\{X[\vartheta, \tau, \lambda]B(\tau, \lambda)\}_m w(\tau, \lambda)\, d\tau \tag{2.2}$$

where $X[\vartheta, \tau, \lambda]$ is the fundamental matrix of Eq. (2.1) with $w \equiv 0$, $X[\tau, \tau, \lambda] = E$ is a unit matrix; l is an arbitrary m-dimensional unit vector, $\|l\| = 1$.

We consider the set $W^\circ(l, \tau, \lambda)$ of program controls $w^\circ(l, \tau, \lambda) \in R$, $t_0 \leqslant t \leqslant \tau \leqslant \vartheta$, satisfying the maximum condition

$$l'\{X[\vartheta, \tau, \lambda]B(\tau, \lambda)\}_m w^\circ(l, \tau, \lambda) = \max_{w \in R} l'\{X[\vartheta, \tau, \lambda]B(\tau, \lambda)\}_m w \tag{2.3}$$

Condition 2.1. For all t, τ and l the set

$$\int_t^\tau B(\zeta, \lambda)\, W^\circ(l, \zeta, \lambda)\, d\zeta$$

is upper-semicontinuous with respect to inclusion as parameter λ varies in a neighborhood of point λ_0.

Lemma 2.1. If the controlled system (2.1) satisfies Condition 2.1 and if the matrices $A(\tau, \lambda)$ and $B(\tau, \lambda)$ are bounded and integrally continuous in parameter λ at point λ_0, then the attainability region $G(\vartheta, t, x, \lambda)$ of system (2.1) is continuous in parameter λ at point λ_0.

Proof. By virtue of Condition 2.1 the limit of any convergent sequence

$$\int_t^{\vartheta} B(\tau, \lambda_k) w^\circ(l, \tau, \lambda_k)\, d\tau, \quad \lambda_k \to \lambda_0, \quad k = 1,2,3, \ldots \tag{2.4}$$

is contained in the set

$$\int_t^{\vartheta} B(\tau, \lambda_0) W^\circ(l, \tau, \lambda_0)\, d\tau$$

for any vector l and any instant $t \geqslant t_0$. From the results in [2, 3] it follows that the fundamental matrix $X[t, \tau, \lambda]$ is continuous in parameter λ at point λ_0, uniformly relative to $t, \tau \in [t_0, \vartheta]$. Consequently, the equality

$$\lim_{\lambda_k \to \lambda_0} \int_t^{\vartheta} l' \{X[\vartheta, \tau, \lambda_k] B(\tau, \lambda_k)\}_m w^\circ(l, \tau, \lambda_k)\, d\tau = \tag{2.5}$$

$$\lim_{\lambda_k \to \lambda_0} \max_{w \in R} \int_t^{\vartheta} l' \{X[\vartheta, \tau, \lambda_k] B(\tau, \lambda_k)\}_m w(\tau, \lambda_k)\, d\tau =$$

$$\max_{w \in R} \int_t^{\vartheta} l' \{X[\vartheta, \tau, \lambda_k] B(\tau, \lambda_k)\}_m w(\tau, \lambda_k)\, d\tau$$

is valid for any convergent sequence (2.4) and for arbitrary l and t. From (2.5) and the continuity of matrix $X[\vartheta, t, \lambda]$ it follows that the support function $\rho[l, \vartheta, t, x, \lambda]$, given by (2.2) of the attainability region $G(\vartheta, t, x, \lambda)$ is continuous in λ at point λ_0.

Using the continuity of the attainability region $G(\vartheta, t, x, \lambda)$ with respect to λ at point λ_0, it is easily verified that the following assertion holds.

Lemma 2.2. If the controlled system (2.1) satisfies Condition 2.1 and if the matrices $A(\tau, \lambda)$ and $B(\tau, \lambda)$ are integrally continuous in parameter λ at point λ_0, then the set

$$\int_t^{\tau} B(\zeta, \lambda) W^\circ(l(\zeta, \lambda), \zeta, \lambda)\, d\zeta$$

is upper-semicontinuous with respect to inclusion as parameter λ varies in a neighborhood of point λ_0 for all t and τ, for any family of unit vector functions $l(\zeta, \lambda)$, continuous in ζ and such that the limit relation $\lim l(\zeta, \lambda) = l(\zeta, \lambda_0)$ as $\lambda \to \lambda_0$ is satisfied uniformly relative to ζ.

3. Let $\rho^{(1)}[l, \vartheta, t, y, \lambda]$ and $\rho^{(2)}[l, \vartheta, t, z, \lambda]$ be the support functions of the attainability regions $G^{(1)}(\vartheta, t, y, \lambda)$ and $G^{(2)}(\vartheta, t, z, \lambda)$ of the pursuing object and pursued object, respectively, by the instant $t = \vartheta$ from the position $y[t] = y$ and $z[t] = z$. We consider the quantity

$$\varepsilon^\circ(t, y, z, \lambda) = \max_{\|l\|=1} \{\rho^{(2)}[l, \vartheta, t, z, \lambda] - \rho^{(1)}[l, \vartheta, t, y, \lambda]\} = \tag{3.1}$$

$$\max_{\|l\|=1} \{l'\{Z[\vartheta, t, \lambda] z\}_m - l'\{Y[\vartheta, t, \lambda] y\}_m +$$
$$\max_{v\in Q} \int_t^{\vartheta} l'\{Z[\vartheta, \tau, \lambda] B^{(2)}(\tau, \lambda)\}_m v(\tau, \lambda) d\tau -$$
$$\max_{u\in P} \int_t^{\vartheta} l'\{Y[\vartheta, \tau, \lambda] B^{(1)}(\tau, \lambda)\}_m u(\tau, \lambda) d\tau\}$$

Definition 3.1. We say that the regular case takes place if a neighborhood $\Omega(\lambda_0)$ of point λ_0 exists such that the maximum in the right-hand side of equality (3.1) is achieved for all $\lambda \in \Omega(\lambda_0)$ on the unity vector $l^\circ(t, y, z, \lambda)$ for all positions $\{t, y, z\}$ that can occur in the game being considered and for which $\varepsilon^\circ(t, y, z, \lambda) > 0$.

We assume that the right-hand sides of Eqs.(1.1) and (1.2) satisfy the following requirements.

Condition 3.1. (1) Matrices $A^{(j)}(t, \lambda)$ and $B^{(j)}(t, \lambda)$, $(j = 1, 2)$ are continuous in t, bounded in λ and integrally continuous in λ at point λ_0. (2) Condition 2.1 is satisfied for Eqs.(1.1) and (1.2).

The validity of the following statements arises form the results in [1, 6] and Lemma 2.1.

Lemma 3.1. If the right-hand sides of Eqs.(1.1) and (1,2) satisfy Condition 3.1, then $\varepsilon^\circ(t, y, z, \lambda)$ of (3.1) is a continuous function of the game position $\{t, y, z\}$ and of parameter λ at point λ_0.

Lemma 3.2. If the right-hand sides of Eqs.(1.1) and (1.2) satisfy Condition 3.1 and if the regular case takes place, then the vector $l^\circ(t, y, z, \lambda)$ depends continuously on the game position $\{t, y, z\}$ and on parameter λ at point λ_0.

The players'optimal strategies $U_\lambda^\circ \div U^\circ(t, y, z, \lambda)$ and $V_\lambda^\circ \div V^\circ(t, y, z, \lambda)$, (see [1]) implying the existence of the saddle point in the encounter game, are determined by the extremal aiming rule, i.e. the sets $U^\circ(t, y, z, \lambda)$ and $V^\circ(t, y, z, \lambda)$ in the region $\varepsilon^\circ(t, y, z, \lambda) > 0$, $t < \vartheta$ consists of all those vectors $u \in P$ and $v \in Q$ that at the instant t satisfy the maximum condition (2.3) with $\tau = t$ and $l = l^\circ(t, y, z, \lambda_0)$; however, if $\varepsilon^\circ(t, y, z, \lambda) = 0$, then $U^\circ(t, y, z, \lambda) = P$ and $V^\circ(t, y, z, \lambda) = Q$. From Lemmas 3.1, 3.2, 2.1 and 2.2 it follows that the optimal strategies U_λ° and V_λ° are integrally semicontinuous in parameter λ at point λ_0 if Condition 3.1 is satisfied and the regular case takes place. Turning to Theorem 1.1, we arrive at the following conclusion.

Theorem 3.1. If the right-hand sides of Eqs.(1.1) and (1.2) satisfy Condition 3.1 and the regular case takes place, then: (1) the value $\varepsilon^\circ(t_0, y^\circ, z^\circ, \lambda)$ of the encounter game problem for objects (1.1) and (1.2) is continuous in parameter λ at point λ_0 and in the initial position $\{t_0, y^\circ, z^\circ\}$ from an arbitrary bounded region Γ in space $\{t, y, z\}$; (2) the set of optimal approximate motions $X^{(\Delta)}(U_\lambda^\circ, V_\lambda^\circ; t_0, y^\circ, z^\circ, \lambda)$ is upper-semicontinuous with respect to inclusion as λ varies in a neighborhood of point λ_0, uniformly with respect to all partitionings Δ of the interval $[t_0, \vartheta]$ and all initial positions $\{t_0, y^\circ, z^\circ\} \in \Gamma$; (3) the set of optimal motions $X(U_\lambda^\circ, V_\lambda^\circ; t_0, y^\circ, z^\circ, \lambda)$ is upper-semicontinuous with respect to inclusion as parameter λ varies in a neighborhood of point λ_0, uniformly with respect to all initial positions $\{t_0, y^\circ, z^\circ\} \in \Gamma$.

4. We consider an encounter problem for objects described by the differential equations

$$y_1^{\cdot} = y_2,\ y_2^{\cdot} = a(t,\lambda)u_1,\ y_3^{\cdot} = y_4,\ y_4^{\cdot} = a(t,\lambda)u_2$$
$$z_1^{\cdot} = z_2,\ z_2^{\cdot} = b(t,\lambda)v_1,\ z_3^{\cdot} = z_4,\ z_4^{\cdot} = b(t,\lambda)v_2$$
$$u_1^2 + u_2^2 \leqslant \mu^2,\ v_1^2 + v_2^2 \leqslant \nu^2,\ \mu > \nu$$
$$a(t,\lambda) = \begin{cases} 1 + a\cos t/\lambda, & \lambda \neq 0 \\ 1, & \lambda = 0, \end{cases} \qquad b(t,\lambda) = \begin{cases} 1 + b\sin t/\lambda, & \lambda \neq 0 \\ 1, & \lambda = 0 \end{cases}$$
$$a = \text{const},\ b = \text{const},\ |a| \leqslant 1,\ |b| \leqslant 1$$

Let the game's payoff $\gamma[\vartheta]$ be determined by the equality

$$\gamma[\vartheta] = [(y_1(\vartheta) - z_1(\vartheta))^2 + (y_3(\vartheta) - z_3(\vartheta))^2]^{1/2}$$

We can verify that the hypotheses of Theorem 3.1 are satisfied for all λ. Carrying out the necessary calculations, we obtain

$$\varepsilon^\circ(t,y,z,\lambda) = \eta - \zeta\frac{(\vartheta - t)^2}{2} + \lambda a\mu\left[(\vartheta - t)\sin\frac{t}{\lambda} + \lambda\left(\cos\frac{\vartheta}{\lambda} - \cos\frac{t}{\lambda}\right)\right] +$$
$$\lambda b\nu\left[(\vartheta - t)\cos\frac{t}{\lambda} - \lambda\left(\sin\frac{\vartheta}{\lambda} - \sin\frac{t}{\lambda}\right)\right]$$
$$l_1^\circ = [x_1 + (\vartheta - t)x_2]/\eta$$
$$l_2^\circ = [x_3 + (\vartheta - t)x_4]/\eta$$
$$\eta = [(x_1 + (\vartheta - t)x_2)^2 + (x_3 + (\vartheta - t)x_4)^2]^{1/2}$$
$$(x_i = y_i - z_i,\ i = 1, 2, 3, 4;\ \zeta = \mu - \nu)$$

The optimal strategies U_λ° and V_λ° are described as follows: (1) if $\varepsilon^\circ(t,y,z,\lambda) > 0$, then the sets $U^\circ(t,y,z,\lambda)$ and $V^\circ(t,y,z,\lambda)$ consist of the single points $u^\circ[t] = \mu l^\circ$ and $v^\circ[t] = \nu l^\circ$, respectively; (2) if $\varepsilon^\circ(t,y,z,\lambda) = 0$, then $U^\circ = P$ and $V^\circ = Q$.

From this example it is easy to ascertain that the requirement of integral continuity with respect to the parameter in the right-hand sides of Eqs.(1.1) and (1.2) is an essential one. In fact, if we assume that $a(t,0) = b(t,0) = k$, where k is an arbitrary constant not equal to unity, then the functions $a(t,\lambda)$ and $b(t,\lambda)$ are not integrally continuous in λ at the point $\lambda = 0$. When $\lambda = 0$ we have

$$\varepsilon^\circ(t,y,z,0) = \eta - {}^1/_2|k|\zeta(\vartheta - t)^2$$

and, consequently, in such a case $\varepsilon^\circ(t,y,z,\lambda)$ is not continuous in λ at the point $\lambda = 0$.

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